Primal-dual characterizations of jointly optimal transmission rate & scheme for distributed sources

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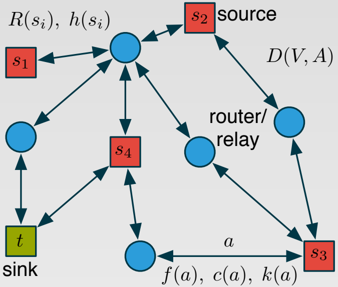
Objective: Losslessly transmit correlated sources to sink over a capacitated network with minimum cost

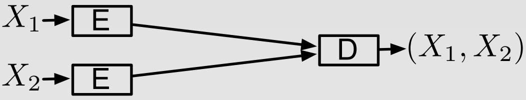


$$\begin{aligned}
\underset{f \geq 0, R}{\text{minimize}} \quad & \sum_{a \in A} k(a) f(a) + \sum_{s \in S} h(s) R(s) & \\
\text{subject to} \quad & f(a) \leq c(a) & a \in A \\
& f(\delta^{in}(v)) - f(\delta^{out}(v)) = 0 & v \in N \\
& R(s) + f(\delta^{in}(s)) - f(\delta^{out}(s)) = 0 & s \in S \\
& R(U) \geq H(X_U | X_{U^c}) & U \subseteq S
\end{aligned}$$

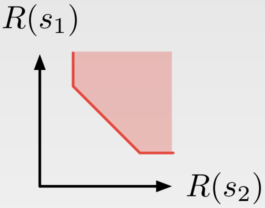
- weighted source cost → $\sum_{s \in S} h(s) R(s)$
- weighted flow cost → $\sum_{a \in A} k(a) f(a)$
- flow feasibility → $f(a) \leq c(a)$ and $f(\delta^{in}(v)) - f(\delta^{out}(v)) = 0$
- flow-rate compatibility → $R(s) + f(\delta^{in}(s)) - f(\delta^{out}(s)) = 0$
- rate feasibility → $R(U) \geq H(X_U | X_{U^c})$

Related work – Slepian and Wolf (1973) through Ramamoorthy (2011)

Slepian and Wolf (1973)[1]

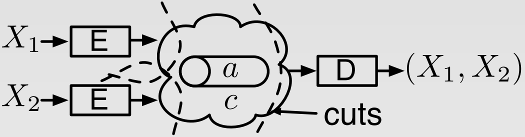


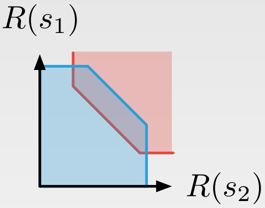
- **Rates for lossless recovery at a single sink using separate encoders**



Related work – Slepian and Wolf (1973) through Ramamoorthy (2011)

Han (1980) [3]

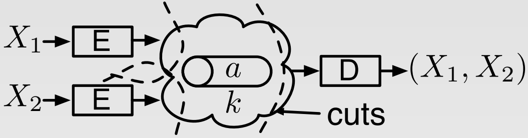


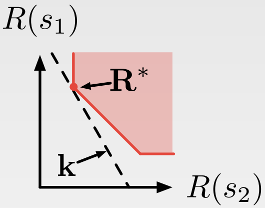


- Rates for lossless recovery at a single sink using separate encoders
- **Over capacitated network**

Related work – Slepian and Wolf (1973) through Ramamoorthy (2011)

Cristescu, Beferull-Lozano, Vetterli (2005) [4]

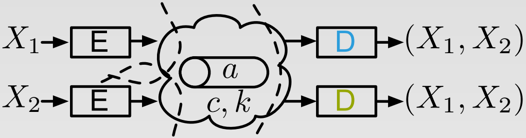


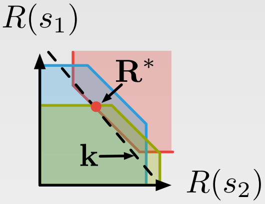


- Rates for lossless recovery at a single sink using separate encoders
- Over ~~capacitated~~ **uncapacitated** network
- **Minimization of nonlinear rate and flow objective over feasible (rate,flow) region**

Related work – Slepian and Wolf (1973) through Ramamoorthy (2011)

Ramamoorthy (2011) [5]





Solution approach: dual decomposition with subgradient descent

- Rates for lossless recovery at ~~single~~ **multiple** sinks (with identical recovery req.) using separate encoders
- Over ~~uncapacitated~~ **capacitated** network
- Minimization of ~~nonlinear~~ **linear** rate and flow objective over feasible (rate,flow) region

. . . and many others

1. **Draper and Wornell (2004)** – achievable lossy coding (Wyner-Ziv) for correlated observations of a single source to a single sink over a sensor network.
2. **Barros and Servetto (2006)** – related formulation / results to Han (1980), pose but don't solve optimization problem over rate region.
3. **Ramamoorthy, Jain, Chou, Effros (2006)** – distributed source coding of multiple sources over network with lossles recovery at multiple receivers (identical recovery req.).
4. **Ho, Médard, Effros, Koetter (2006)** – RNLC to multicast, identifies RLNC error exponents as natural extensions of SW error exponents.
5. **Han (2011)** – extends Han (1980) from one to multiple sinks (identical recovery req.).

. . . and many, many more

Summary of results

$$\begin{aligned}
\underset{f\geq 0, R}{\text{minimize}} \quad & \sum_{a\in A} k(a)f(a) + \sum_{s\in S} h(s)R(s) && \\
\text{subject to} \quad & f(a) \leq c(a) && a \in A \\
& f(\delta^{in}(v)) - f(\delta^{out}(v)) = 0 && v \in N \\
& R(s) + f(\delta^{in}(s)) - f(\delta^{out}(s)) = 0 && s \in S \\
& R(U) \geq H(X_U | X_{U^c}) && U \subseteq S
\end{aligned}$$

weighted flow cost

weighted source cost

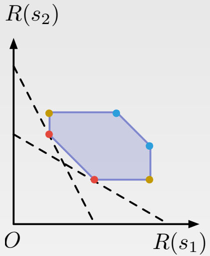
flow feasibility

flow-rate compatibility

rate feasibility

Key Results

1. Structure of set of feasible rates $\mathcal{R}$
2. Active and inactive constraints
3. Optimal primal-dual variables from reduced costs



Result 1: Structure of set of feasible rates $\mathcal{R}$

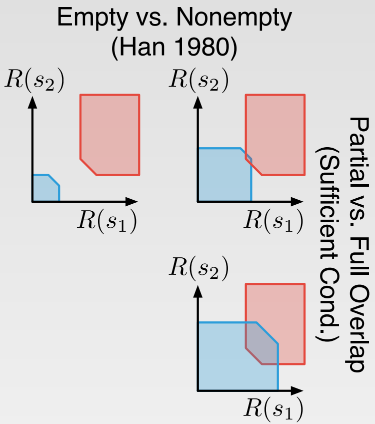
Context:

- Han (1980) matching condition characterizes **empty vs. non-empty** $\mathcal{R}$

$$H(X_U | X_{U^c}) \leq \rho_c(U), \;\; U \subseteq S.$$

- Our result: sufficient condition for **full vs. partial overlap** $\mathcal{R}$

$\mathcal{R}$ is overlap of supportable & achievable rate regions



Proposition

Frank & Tardos (1988) [6] cross inequality implies rate region intersection contains both base polytopes.

$$H(X_U \mid X_{U^c}) - H(X_{U \setminus T} \mid X_{U^c \cup T}) \leq \rho_c(T) - \rho_c(T \setminus U), \forall T, U \subseteq S.$$

Result 1: Structure of set of feasible rates (cont.)

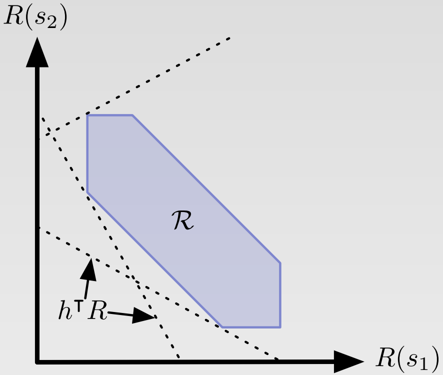
Cross inequality satisfied $\Rightarrow \mathcal{R}$ is a **generalized polymatroid**, and:

Proposition

- *Extreme points of $\mathcal{R}$ are known (since $\mathcal{R}$ is projection of a polymatroid)*
- *The LP*

$$\min_{R \in \mathcal{R}} \sum_{s \in S} h(s) R(s)$$

has an explicitly characterized solution (via Edmonds 1970)

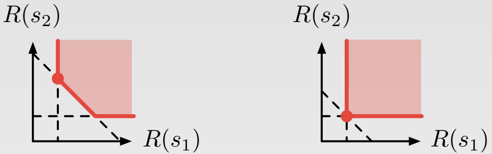


Takeaway: cross inequality $\Rightarrow$ LP soln. is SW vertex; network capacities nonbinding.

$$\mathcal{R} = \{R : H(X_U | X_{U^c}) \leq R(U) \leq \rho_c(U),\ U \subseteq S\}$$

Result 2: Active & inactive constraints

- Slepian-Wolf gives *half-space* representation of polyhedral rate region
- Greedy algorithm establishes bijective map between source permutations π and *vertices* R_π of $\mathcal{R}$ (Edmonds 1970) [7]
- A *degenerate* vertex has $> |S|$ active inequalities



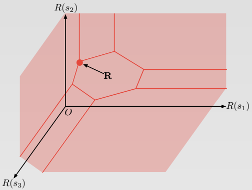
non-degenerate vertex (2 active constraints) | degenerate vertex (3 active constraints)

Q: for which $U \subseteq S$ will SW constraint $R(U) \geq H(X_U | X_{U^c})$ be (in)active at vertex R_π of $\mathcal{R}_{SW}$?

A: tightness of each SW constraint at each vertex is determinable from π and the source conditional independence (C.I.) structure.

Result 2: Which $R(U)$ are tight at vertex R_π of $\mathcal{R}_{SW}$?

Fix perm. $\pi = (1,2,3)$; let $R_\pi = (H(X_1 \mid X_2, X_3), H(X_2 \mid X_3), H(X_3))$ be extreme point in $\mathcal{R}_{SW}$.



π gives 3 necessarily active constraints at R_π:

$$R_\pi(1) = H(X_1 \mid X_2, X_3)$$
$$R_\pi(1) + R_\pi(2) = H(X_1, X_2 \mid X_3)$$
$$R_\pi(1) + R_\pi(2) + R_\pi(3) = H(X_1, X_2, X_3)$$

But there may be additional active const. at R_π.

Proposition

Given arbitrary π and arbitrary $U \subseteq S$:

$$R_\pi(U) = H(X_U | X_{U^c}) \Leftrightarrow (X_{U \setminus U_{k_j - 1}} \perp X_{U_{k_j-1} \setminus U_{k_{j-1}}}) | X_{U^c_{k_j} \setminus U}, \; j = 1, \ldots, m.$$

where $U = (k_1, \ldots, k_m)$ and $U_i = (k_1, \ldots, k_i)$ are ordered by π.

Example: $R_\pi(2) = H(X_2 | X_1, X_3)$ iff $(X_2 \perp X_1) | X_3$.

Takeaway: constraint U tightness at R_π determined by π & CI structure.

Result 3: Optimal primal-dual vars. via reduced costs

Problem: "direct" solution techniques not computationally feasible:

- *Exhaustive direct search:* evaluate cost at each of $|S|!$ rate vertices
- *Primal-dual approach:* find (P,D) variables that are (P,D) feasible and satisfy complementary slackness – $|V| + |A| + 2^{|S|}$ dual vars.

Proposition

A feasible vertex R_π and associated min-cost flow f_π^* is optimal if there exists $z : V \to \mathbb{R}$ and reduced costs satisfying

$$\bar{k}(a) < 0 \implies f_\pi^*(a) = c(a) \quad \bar{k}(a) > 0 \implies f_\pi^*(a) = 0$$

and $\bar{h}(s_{\pi(1)}) \geq \bar{h}(s_{\pi(2)}) \geq \cdots \geq \bar{h}(s_{\pi(n)}) \geq 0$, *where*

$$\bar{k}(a) \equiv k(a) - (z(head(a)) - z(tail(a))), \ a \in A, \quad \bar{h}(s) \equiv h(s) - z(s), \ s \in S.$$

Value: potential on $|V|$ nodes builds $|V| + |A| + 2^{|S|}$ (P,D) solutions.

Conclusion

1. **Goal:** leverage the combinatorial structure of the contrapolymatroid achievable rate region and the polymatroid supportable rate region to provide explicit solutions for (flow,rate) linear programs.

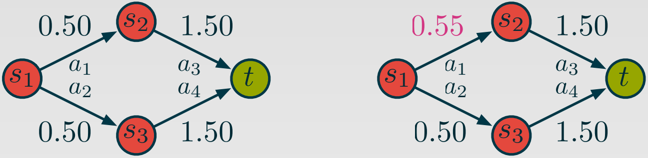
2. **Current (partial) results:**
   1. structure of feasible set of rates
   2. identify (in)active constraints at each rate vertex
   3. efficient characterization of optimality via primal-dual and reduced costs

3. **Extensions:** even more explicit characterizations of feasible set and LP solutions

Result 3: An example

arc costs $k(a_1) = 1.0 \quad k(a_2) = 2.0 \quad k(a_3) = 2.0 \quad k(a_4) = 4.0$

source costs $h(s_1) = 2.00 \quad h(s_2) = 1.25 \quad h(s_3) = 0.50$



$f^* = \begin{bmatrix} 0.5 & 0.01 & 1.48 & 0.67 \end{bmatrix} \quad f^* = \begin{bmatrix} 0.51 & 0.00 & 1.49 & 0.66 \end{bmatrix}$

$$R^* = \begin{bmatrix} 0.51 & 0.98 & 0.66 \end{bmatrix}$$

vertex of SW region for s_1, s_3, s_2

$\bar{k} = \begin{bmatrix} -3 & 0 & 0 & 0 \end{bmatrix}$ ←reduced arc costs→ $\bar{k} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

$\bar{h} = \begin{bmatrix} 8 & 1.25 & 4.5 \end{bmatrix}$ ←reduced source costs→ $\bar{h} = \begin{bmatrix} 5 & 1.25 & 4.5 \end{bmatrix}$

References I

- D. Slepian and J. K. Wolf, "Noiseless coding of correlated information sources," *IEEE Trans. Inf. Theory*, vol. 4, pp. 471–480, 1973.
- T. M. Cover, "A proof of the data compression theorem of Slepian and Wolf for ergodic sources (corresp.)," *IEEE Trans. Inf. Theory*, vol. 21, no. 2, pp. 226–228, 1975.
- T. S. Han, "Slepian-Wolf-Cover theorem for networks of channels," *Information and Control*, vol. 47, no. 1, pp. 67–83, 1980.
- R. Cristescu, B. Beferull-Lozano, and M. Vetterli, "Networked Slepian-Wolf: theory, algorithms, and scaling laws," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4057–4073, 2005.
- A. Ramamoorthy, "Minimum cost distributed source coding over a network," *IEEE Trans. Inf. Theory*, vol. 57, no. 1, pp. 461–475, Jan. 2011.

References II

A. Frank and É. Tardos, "Generalized polymatroids and submodular flows," *Mathematical Programming*, vol. 42, no. 1–3, 1988.

A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*. Springer, 2003.