Primal-Dual Characterizations of Jointly Optimal Transmission Rate and Scheme for Distributed Sources

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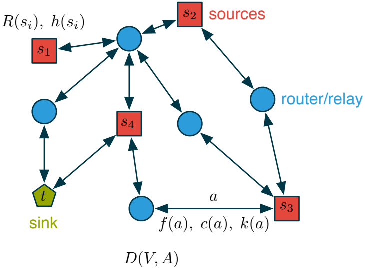
Data Compression Conference
Snowbird, UT
March 27th, 2014

Outline

Objective

Lossless transmission of correlated sources to sink over *capacitated* network with minimum cost



flow cost   source cost

$$\underset{f \geq 0, R}{\text{minimize}} \quad \sum_{a \in A} k(a) f(a) + \sum_{s \in S} h(s) R(s)$$

subject to

$$f(a) \leq c(a) \qquad a \in A$$

flow feasibility → $$f(\delta^{in}(v)) - f(\delta^{out}(v)) = 0 \qquad v \in N$$

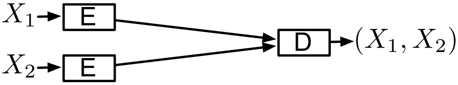
flow supports rate → $$R(s) + f(\delta^{in}(s)) - f(\delta^{out}(s)) = 0 \qquad s \in S$$

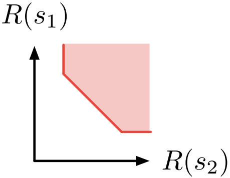
rate feasibility → $$R(U) \geq H(X_U | X_{U^c}) \qquad U \subseteq S$$

Related Work

Slepian & Wolf (1973) through Ramamoorthy (2011)

Slepian and Wolf (1973) [1]



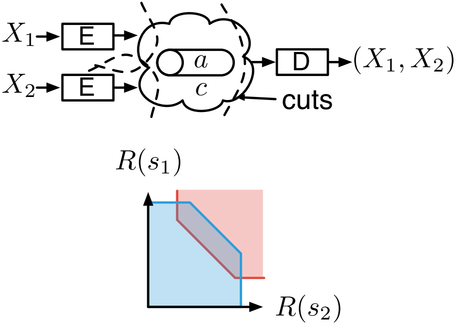


- **Rates for lossless recovery at a single sink using separate encoders**

Related Work

Slepian & Wolf (1973) through Ramamoorthy (2011)

Han (1980) [2]

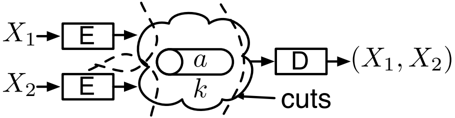


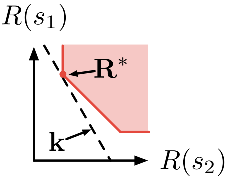
- Rates for lossless recovery at a single sink using separate encoders
- **Over capacitated network**

Related Work

Slepian & Wolf (1973) through Ramamoorthy (2011)

Cristescu, Beferull-Lozano, Vetterli (2005) [3]



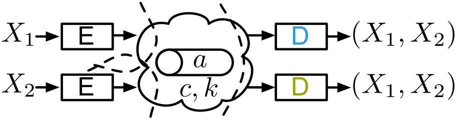


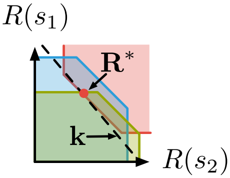
- Rates for lossless recovery at a single sink using separate encoders
- Over ~~capacitated~~ **uncapacitated** network
- **Minimization of nonlinear rate and flow objective over feasible (rate, flow) region**

Related Work

Slepian & Wolf (1973) through Ramamoorthy (2011)

Ramamoorthy (2011) [4]





Solution approach: dual decomposition with subgradient descent

- Rates for lossless recovery at ~~single~~ **multiple** sinks (w/ identical recovery req.) using separate encoders
- Over ~~uncapacitated~~ **capacitated** network
- Minimization of ~~nonlinear~~ **linear** rate and flow objective over feasible (rate, flow) region

Related Work

. . . & Many Others

1. **Draper and Wornell (2004)**—achievable lossy coding (Wyner-Ziv) for correlated observations of a single source to a single sink over a sensor network
2. **Barros and Servetto (2006)**—related formulation/results to Han (1980), pose but don't solve optimization problem over rate region
3. **Ramamoorthy, Jain, Chou, Effros (2006)**—distributed source coding of multiple sources over network w/ lossless recovery at multiple receivers (identical recovery req.)
4. **Ho, Médard, Effros, Koetter (2006)**—RLNC to multicast, identifies RLNC error exponents as natural extensions of SW error exponents
5. **Han (2011)**—extends Han (1980) from one to multiple sinks (identical recovery req.)

. . . and many, many more

Summary of Results

flow cost | source cost

$$\underset{f \geq 0, R}{\text{minimize}} \quad \sum_{a \in A} k(a) f(a) + \sum_{s \in S} h(s) R(s)$$

subject to

flow feasibility → $f(a) \leq c(a) \quad a \in A$

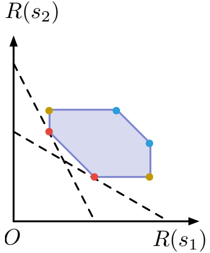
flow feasibility → $f(\delta^{in}(v)) - f(\delta^{out}(v)) = 0 \quad v \in N$

flow supports rate → $R(s) + f(\delta^{in}(s)) - f(\delta^{out}(s)) = 0 \quad s \in S$

rate feasibility → $R(U) \geq H(X_U | X_{U^c}) \quad U \subseteq S$

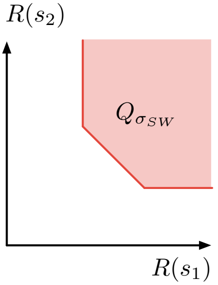
Key Results

1. Structure of set of feasible rates $\mathcal{R}$
2. Active & inactive constraints
3. Optimal primal-dual variables from reduced costs



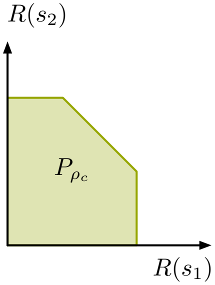
Outline

Achievable & Supportable Rates



Slepian-Wolf (1973)

- Set of *achievable* rates
- Contrapolymatorid associated w/ $\sigma_{SW}(U) = H(X_U \mid X_{U^c})$



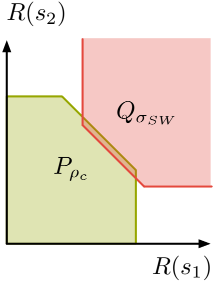
Meggido (1974) [5]

- Set of *supportable* rates
- Polymatroid associated w/ $\rho_c(U) = c(\text{min-cut}(U))$

Bijective map between source permutations π and vertices R_π of (contra)polymatroids (Edmonds 1970) [6]

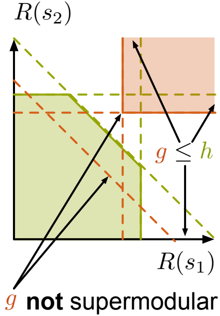
Feasible Rates

Han (1980)



- Intersection non-empty iff $\sigma_{SW}(U) \leq \rho_c(U)$
- Achievability & converse proofs for $R \in \mathcal{R}$

Sufficiency Example



$g(U) \leq h(U)$

- necessary
- w/o sub-/supermodularity is **not** sufficient

Efficient Transmission of Sources to Sink

flow cost | source cost

$$\begin{aligned} \underset{f \geq 0, R}{\text{minimize}} \quad & \sum_{a \in A} k(a) f(a) + \sum_{s \in S} h(s) R(s) \\ \text{subject to} \quad & f(a) \leq c(a) & a \in A \\ & f(\delta^{in}(v)) - f(\delta^{out}(v)) = 0 & v \in N \\ & R(s) + f(\delta^{in}(s)) - f(\delta^{out}(s)) = 0 & s \in S \\ & R(U) \geq H(X_U | X_{U^c}) & U \subseteq S \end{aligned}$$

- flow feasibility → $f(a) \leq c(a)$ and $f(\delta^{in}(v)) - f(\delta^{out}(v)) = 0$
- flow supports rate → $R(s) + f(\delta^{in}(s)) - f(\delta^{out}(s)) = 0$
- rate feasibility → $R(U) \geq H(X_U | X_{U^c})$

- Linear program with $|A| + |V| - 1 + 2^{|S|}$ inequalities
  - If $|S| = \mathcal{O}(|V|)$, then the LP is exponential in the size of the graph
- Optimal solution (f^*, R^*) will satisfy $R^*(S) = H(X_S)$ [2]

Outline

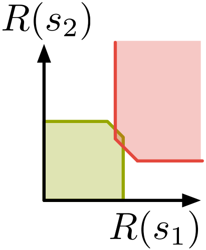
Partial vs. Full Overlap

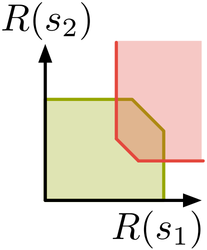
Context:

- Han (1980) characterizes **empty vs. non-empty** $\mathcal{R}$
- **Q:** When are all efficient vertices retained in the intersection?

Partial vs. Full Overlap

(Sufficient Condition)





A: A sufficient condition for **full vs. partial overlap** $\mathcal{R}$

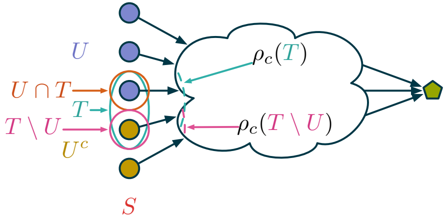
Cross Inequality

Proposition

Frank & Tardos (1988) [7] cross inequality implies rate region intersection contains both base polytopes.

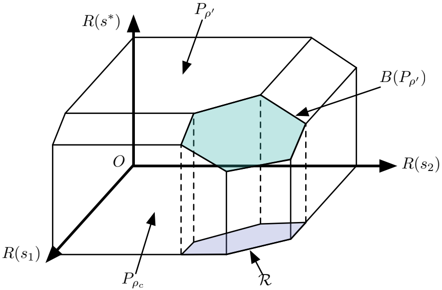
$$H(X_{U \cap T} \mid X_{U^c}) \leq \rho_c(T) - \rho_c(T \setminus U), \forall T, U \subseteq S$$

Cross inequality specialzed to conditional entropy & min-cut capacity



$$H(X_{U \cap T} \mid X_{U^c}) \leq \rho_c(T) - \rho_c(T \setminus U)$$

Full Overlap & Generalized Polymatroids



Fujishige (2005) [8]

- Cross-inequality satisfied $\Rightarrow \mathcal{R}$ is a **generalized polymatroid**
- $\mathcal{R}$ is projection of a base polytope of polymatroid

Full Overlap & Generalized Polymatroids

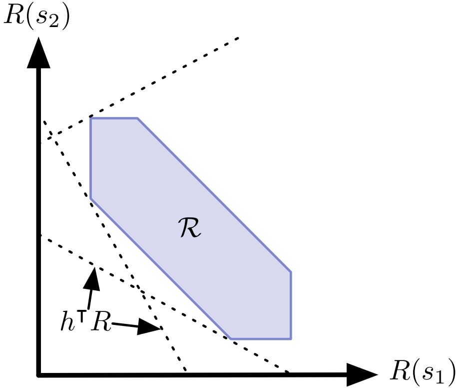
Proposition

Satisfying cross inequality $\Rightarrow$

- *Extreme points of* $\mathcal{R}$ *are known*
- *The LP*

$$\min_{R \in \mathcal{R}} \sum_{s \in S} h(s) R(s)$$

has an explicitly characterized solution (via Edmonds 1970)



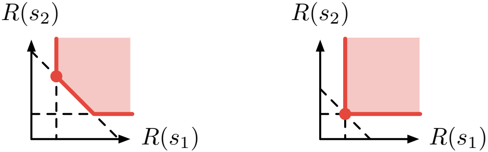
Takeaway: Cross inequality $\Rightarrow$ soln. is SW vertex; network capacities non-binding

$$\mathcal{R} = \{R : H(X_U | X_{U^c}) \leq R(U) \leq \rho_c(U),\ U \subseteq S\}$$

Active & Inactive Constraints

Polyhedral rate region

- Slepian & Wolf gives *half-space* representation
- Greedy algorithm gives *vertex* representation (Edmonds 1970) [6]
- A *degenerate* vertex has $> |S|$ active inequalities



non-degenerate vertex (2 active constraints)

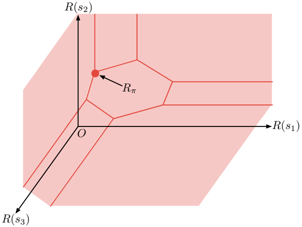
degenerate vertex (3 active constraints)

Q: For which $U \subseteq S$ will SW constraint $R(U) \geq H(X_U|X_{U^c})$ be (in)active at vertex R_π of $\mathcal{R}_{SW}$?

A: Tightness of each SW constraint at each vertex is determinable from π and the source conditional independence (C.I.) structure

Which $R(U)$ are tight at vertex R_π of $\mathcal{R}_{SW}$?

Fix perm. $\pi = (1,2,3)$; let $R_\pi = (H(X_1 \mid X_2, X_3), H(X_2 \mid X_3), H(X_3))$



π gives 3 necessarily active constraints at R_π:

$$R_\pi(1) = H(X_1 \mid X_2, X_3)$$
$$R_\pi(1) + R_\pi(2) = H(X_1, X_2 \mid X_3)$$
$$R_\pi(1) + R_\pi(2) + R_\pi(3) = H(X_1, X_2, X_3)$$

But there may be additional active const. at R_π.

Proposition

Given π and $U = \{s_{k_1}, \ldots, s_{k_m}\} \subseteq S$, let $U_i = \{s_1, \ldots, s_i)$; then

$$R_\pi(U) = H(X_U | X_{U^c}) \Leftrightarrow (X_{U \setminus U_{k_j - 1}} \perp X_{U_{k_j-1} \setminus U_{k_{j-1}}}) | X_{U^c_{k_j} \setminus U}, \; j = 1, \ldots, m.$$

Example: $R_\pi(2) = H(X_2 | X_1, X_3)$ iff $(X_2 \perp X_1) | X_3$

Takeaway: Constraint U tightness at R_π determined by π & CI structure.

Outline

Dual LP

Problem: "direct" solution techniques not computationally feasible:

- *Exhaustive direct search:* evaluate cost at each of $|S|!$ rate vertices
- *Primal-dual approach:* find (P,D) variables that are (P,D) feasible and satisfy complementary slackness—$|V| + |A| + 2^{|S|}$ dual vars.

$$\begin{aligned} \underset{x \le 0, y \ge 0, z}{\text{maximize}} \quad & \sum_{a \in A} c(a)x(a) + \sum_{U \subseteq S} H(X_U | X_{U^c}) y_U \\ \text{subject to} \quad & x(a) + z(\text{head}(a)) - z(\text{tail}(a)) \le k(a) \quad && a \in A \\ & \sum_{U \ni s} y_U + z(s) - z(t) = h(s) \quad && s \in S \end{aligned}$$

1. $x(a) \Rightarrow$ flow under capacity
2. $z(v) \Rightarrow$ conservation of flow at node
3. $y_U \Rightarrow$ Slepian-Wolf feasibility

Reduced Costs

Let

$$\bar{k}(a,z) \triangleq k(a) - (z(\text{head}(a)) - z(\text{tail}(a)))$$
$$\bar{h}(s,z) \triangleq h(s) - (z(s) - z(t))$$

and rewrite

$$\begin{aligned} \underset{x\leq 0, y\geq 0, z}{\text{maximize}} \quad & \sum_{a\in A} c(a)x(a) + \sum_{U\subseteq S} H(X_U|X_{U^c})y_U \\ \text{subject to} \quad & x(a) \leq \bar{k}(a,z) && a\in A \\ & \sum_{U\ni s} y_U = \bar{h}(s,z) && s \in S \end{aligned}$$

Obsv: $x^*(a) = \min(0, \bar{k}(a,z^*))$—we can eliminate $|A|$ of the dual variables **Q:** Can we do the same with y_U? **A:** Yes!

Optimal Primal-Dual Vars. via Reduced Costs

Recall: Inactive const. $\Rightarrow y_U^* = 0$
Source permutation π gives $|S|$ necessarily active const. at R_π

Proposition

A feasible vertex R_π and associated min-cost flow f_π^* is optimal if there exists $z : V \to \mathbb{R}$ and reduced costs satisfying

$$\bar{k}(a) < 0 \implies f_\pi^*(a) = c(a) \quad \bar{k}(a) > 0 \implies f_\pi^*(a) = 0$$

and $\bar{h}(s_{\pi(1)}) \geq \bar{h}(s_{\pi(2)}) \geq \cdots \geq \bar{h}(s_{\pi(n)}) \geq 0.$

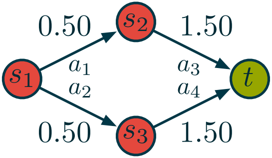
Value: Potential on $|V|$ nodes builds $|V| + |A| + 2^{|S|}$ (P, D) solutions.

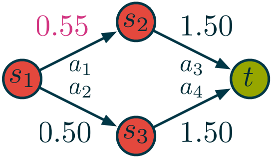
Optimal Primal-Dual Vars. via Reduced Costs

An Example

arc costs $k(a_1) = 1.0 \quad k(a_2) = 2.0 \quad k(a_3) = 2.0 \quad k(a_4) = 4.0$

source costs $h(s_1) = 2.00 \quad h(s_2) = 1.25 \quad h(s_3) = 0.50$





$f^* = \begin{bmatrix} 0.5 & 0.01 & 1.48 & 0.67 \end{bmatrix}$ $\quad f^* = \begin{bmatrix} 0.51 & 0.00 & 1.49 & 0.66 \end{bmatrix}$

$$R^* = \begin{bmatrix} 0.51 & 0.98 & 0.66 \end{bmatrix}$$

vertex of SW region for s_1, s_3, s_2

$\bar{k} = \begin{bmatrix} -3 & 0 & 0 & 0 \end{bmatrix}$ ⟵ reduced arc costs ⟶ $\bar{k} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

$\bar{h} = \begin{bmatrix} 8 & 1.25 & 4.5 \end{bmatrix}$ ⟵ reduced source costs ⟶ $\bar{h} = \begin{bmatrix} 5 & 1.25 & 4.5 \end{bmatrix}$

Outline

Conclusion

1. **Goal:** Leverage the combinatorial structure of the contrapolymatroid achievable rate region and the polymatroid supportable rate region to provide explicit solutions for (flow,rate) linear programs.
2. **Current (partial) results:**
    1. Structure of feasible set of rates
    2. Identify (in)active constraints at each rate vertex
    3. Efficient characterization of optimality via primal-dual and reduced costs
3. **Extensions:** Even more explicit characterizations of feasible set and LP solutions

Acknowledgments

Supported by the AFOSR under agreement number FA9550-12-1-0086

Outline

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